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SOME STRUCTURAL AND DYNAMICAL PROPERTIES OF MANDELBROT SET

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ABSTRACT

In this paper, we have done few investigations on some dynamical as well as structural properties of the Mandelbrot set, which arises as a fractal from the iteration of the complex polynomial of the form $z^2 + c$. We have also discussed about some amazing features shown by the periodic numbers and rotation numbers related to the primary bulbs of the Mandelbrot set.

KEYWORDS: Critical Point, Julia Set, Mandelbrot Set, Fixed Point, Periodic Point, Primary Bulb, Iteration of a Map, Rotation Number, Schwarzian Derivative

1. INTRODUCTION

The Mandelbrot set has a celebrated place in fractal geometry, a field first investigated by the French Mathematicians, Gaston Julia and Pierre Fatou, as a part of complex dynamics in the beginning of the 20th century. Gaston Julia (1893-1978) wrote a paper titled, "*M´emoire sur l'iteration des fonctions rationelles*" (A Note on the Iteration of Rational Functions) **[**19], where, he first introduced the modern idea of a Julia set as a part of complex dynamics. In this paper, Julia gave a precise description of the set of those points of the complex plane, whose orbits under the iteration of a rational function stayed bounded. In 1978, Robert W. Brooks and Peter Matelski investigated some subgroups of Kleinian groups [27] and, as a part of this investigation; they first introduced the concept of what we now called Mandelbrot set.

Benoit Mandelbrot (1924-2010) was a Polish-born French mathematician, who spent most of his career at IBM's Thomas J. Watson Research Center in Yorktown Height, New York. He was inspired by Julia's above mentioned paper on complex dynamics and used computers to explore these works. In the year 1977, as a result of his research, he discovered one of the most famous fractals, which now bears his name: the Mandelbrot set. On 1st March 1980, Mandelbrot first visualized this set [28]. He studied the parameter space of the complex quadratic polynomials in an article, which appeared in the '*Annals of New York Academy of science*' [22].

The Mathematical study of the Mandelbrot set actually began with the works of Adrien Douady and John H. Hubbard [13], who established many of its fundamental properties and named the set in honor of Mandelbrot. Interest in the subject flourished over, and many other well known mathematicians began to study the Mandelbrot set. Heinz-Otto Peitgen and Peter Richter are the names of two such mathematicians who became well known for promoting the Mandelbrot set with computer oriented graphics and books [25].

A good account of developing period of the theory of complex dynamics can be found in [4], [8], [9], [31]. The authors are among the most active contributors to this field.

Mandelbrot set may well be one of the most familiar images produced by the mathematicians and other related scientists of the 20th century. It challenges the familiar notion that the domain of simplicity and complexity are entirely different. Because, the mathematical formula that is involved in the construction of Mandelbrot set consists of simple operations like multiplication and addition, still it produces a shape of great organic beauty and complexity with infinite subtle variations. The developments arising from the Mandelbrot set have been as diverse as the alluring shapes it generates.

The shape of the Galaxies broke all Euclidean laws of the man-made world and deferred from the properties of natural world. If one identified an essential structure like this, Mandelbrot claimed, that the concept of Mandelbrot set, in general fractal geometry, could be applied to understand its component parts and make postulations about what it will become in future. For instance, interested readers may see [29] for study about distribution of galaxies in an observed universe. In today's world of wireless communication, many wireless devices use fractal based compact and potable antennas that pick up the widest range of known frequencies [2], [20]. Fractal art is a form of algorithmic art created by fractal objects produced by repeated iterations of some mathematical rules and representing the calculated results as still images, animations etc. The Mandelbrot set can be considered as a great icon for fractal art. Graphic design and image editing programs use fractal to create beautifully complex landscapes and life-like special effects. Interested readers can go through [5], [26] for finding such applications. Fractal statistical analyses of forest can measure and quantify how much carbon dioxide the world can safely process [23]. Fractal geometry may also be applied to the various fields of medicine such as cardiovascular system, neurobiology, pathology and molecular biology [7], [17].

The Mandelbrot set, like most of the other fractals, arises from a simple iterative process. The process involved here is the iteration of the non-linear relation $z_{n+!} = z_n^{\ 2} + c$ on the points of the complex plane. It turns out that the same relation was already studied in the early 20th century by French Mathematicians Gaston Julia and Pierre Fatou which lead to the discovery of the Julia sets. Like the Mandelbrot set, the Julia set also have a fractal structure and are generated by using the same iterative process employed in the generation of the Mandelbrot set, but with slightly different initial conditions. Interested reader may go through [21]. There is only one Mandelbrot set and infinitely many Julia sets- each point on the complex plane acting as a parameter to the Julia set.

The Mandelbrot set, a very beautiful fractal structure enjoys a special status as a cultural icon. Also, deep mathematics underlies the Mandelbrot set. Despite years of study by brilliant mathematicians, some natural and simple-to-state questions still remains un-answered. For example, though Mandelbrot set was known to the mathematical community since 1977, due to the complex form of shape, its area was estimated approximately to be $1.0565918849 \pm 0.0000000028$ by Thorsten Förstemann, only in 2012 [16]. Much of the re-birth of interest in complex dynamics was motivated by efforts to understand the stunning images of Mandelbrot set, which is the prime objective of this paper.

The rest of the paper is organized as follows: in section 2, we provide a review of preliminary concepts and definitions. In section 3, we have described about the geometrical structure and different components related to the Mandelbrot set, which are required for our further investigation. Section 4 contains discussion on some structural as well as dynamical properties of the Mandelbrot set and, some amazing features shown by the periodic numbers and rotation numbers are related to the primary bulbs of the Mandelbrot set. Finally, in section 5, we have given a concluding remark of our study.

2. PRELIMINARIES

In order to carry out the study, we first need to provide some definitions concerning classical deterministic chaotic dynamical systems that are discussed in this section.

Definition 2.1 [18]

The *orbit* of a number z_0 under function $f:\hat{C} \to \hat{C}$ where $\hat{C}$ denote the extended complex plane i.e., $\hat{C} = C \cup \{\infty\}$ is defined as the sequence of points

$$z_0,\ z_1 = f(z_0), z_2 = f^2(z_0), \cdots, z_n = f^n(z_0) = f(z_{n-1}), \qquad \cdots \tag{2.1}$$

Here, f^n denote the nth iterate of f, that is, f composed with itself n times. The point z_0 is called the seed of the orbit.

For each point $z_0 \in \hat{C}$, we are interested in the behavior of the sequence given in (2.1) and in particular, what happens as n goes to infinity.

Definition 2.2 [14]

A point $z_0 \in \hat{C}$ is called *periodic point* of f if $f^n(z_0) = z_0$ for some integer $n \geq 1$. The smallest n with this property is called the period of z_0. Thus, the periodic points of z_0 are the zeros of the function $F(z_0, f) = f^n(z_0) - z_0$.

A periodic point with period one is termed as *fixed point* of f i.e., z_0 is a *fixed point* of f if $f(z_0) = z_0$.

Definition 2.3: [18]

Let $v \in \hat{C}$. For any complex valued function $f: D \to D$ where $D \subseteq \hat{C}$, the *attracting basin* or *basin of attraction* of v under the function f is defined as the set $A_f(v)$ of all seed values whose orbit limits to the point v, i.e.

$$A_f(v) = \{z \in D:\ f^n(z) \to v\}$$

Definition 2.4 [12]

The *multiplier* (or *eigenvalue*, *derivative*) λ of a rational map f iterated n times, at the periodic point z_0 is defined as:

$$\lambda = \begin{cases} f^{n'}(z_0), & if \ z_0 \neq \infty \\ \dfrac{1}{f^{n'}(z_0)}, & if \ z_0 = \infty \end{cases}$$

where $f^{n'}(z_0)$ is the first derivative of f^n with respect to z at z_0.

Note that, the multiplier is same at all periodic points of a given orbit. Therefore, it can be regarded as multiplier of the periodic orbit.

The absolute value of the multiplier is called the *stability index* of the periodic point. It is used to check the stability of periodic points.

Definition 2.5 [12]

A periodic point z_0 is called attracting periodic point if $|\lambda| < 1$, supper attracting if $|\lambda| = 0$ and is repelling if $|\lambda| > 1$. It is called indifferent or neutral when $|\lambda| = 1$

A dynamical system is a collection of three things $\langle X, T, F \rangle$ where, X is a state space, T is a set of time and $F : X \times T \to X$ is a function that specifies how the state evolves with time, i.e. $F(x,t)$ give the state at time t when the initial state was x.

Definition 2.6 [6]

Let X be a metric space and suppose $f : X \to X$ is continuous then the periodic points of f are called *dense* in X if for any periodic point p_1 of f and for any $\varepsilon > 0$, however small may be, the open sphere $S_\varepsilon(p_1)$ contains another periodic point p_2 of f.

Definition 2.7 [6]

A continuous map $f : X \to X$ on the metric space X is called *transitive* if for any open sets U and V of X, there exists a natural number n such that $f^n(U) \cap V \neq \phi$.

Definition 2.8 [6]

Let X be a metric space and suppose $f : X \to X$ is a continuous map, then we say that f exhibits *sensitive dependence on initial conditions* if there exist a constant $\beta > 0$ such that for any $p \in X$ and any neighborhood U of p, there exists $n > 0$ and $y \in U$ such that $d\left[f^n(p), f^n(y)\right] > \beta$

Definition 2.9 [6]

In a dynamical system $\langle X, T, f \rangle$ the function f is called *chaotic* on X if the following two conditions are fulfilled:

1. Periodic points for f are *dense* in X,
2. The function f is *transitive* on X.

Note that according to most of the literature, f is considered to be *chaotic* on X if together with these two conditions it also satisfied:

3. f depends *sensitively on initial conditions*.

Devaney proved that conditions (1) & (2) together implies condition (3), proof of this result may be found in [3].

To carry out our study for the rest of this paper we consider the maps of the form:

$$f_c(z) = z^2 + c \quad \textbf{(2.2)}$$

For different values of the parameter $c \in \hat{C}$.

Definition 2.10 [6]

The *Julia set* of f_c, denoted by $J(f_c)$, is the set of all points at which f_c exhibits sensitive dependence on initial condition. That is, $J(f_c)$ is the chaotic set for f_c. The complement of $J(f_c)$ is a stable set for f_c and it is called *Fatou set* of f_c.

Definition 2.11 [6]

The set $K(f_c)$ of all those points of $\hat{C}$ which do not converge to ∞ under iteration of the map f_c is called the *filled in Julia set* of the map f_c.

Theorem 2.1

The filled in Julia set $K(f_c)$ is contained inside the closed disc of radious $\max\{|c|, 2\}$. That is $K(f_c) \subseteq \{z : |z| \le \max\{|c|, 2\}\}$.

The proof of this theorem can be found in [8]

Definition 2.12 [24]

A point $z_0 \in \hat{C}$ is called critical point for the analytic function f if $f'(z_0) = 0$.

The simplest method for visualizing Julia sets is 'Escape Time' algorithm. For the details of how the Julia set can be visualized interested reader may see [21]. The computation of Julia sets for various values of the parameter 'c' shows that as the value of the parameter changes, a dramatic change in the shape of the Julia set takes place. At this point, a natural question arises - how to classify or understand all of these interesting shapes of Julia sets? The structure of the Julia set is strongly influenced by the behavior of the critical point of f_c. To know the role of the critical point, we need to first explore the concept of Schwarzian derivative.

Definition 2.13 [8]

The *Schwarzian derivative* of a function f is defined as

$$Sf(x) = \frac{D^3 f(x)}{Df(x)} - \frac{3}{2}\left(\frac{D^2 f(x)}{Df(x)}\right)^2$$

Where, $D^n f(x)$ represent the n[th] derivative of the function $f(x)$, $n=1,2,3$.

Theorem 2.2 [8]

Suppose that Sf is always negative. If x_0 is an attracting periodic points of f, then either the immediate basin of attraction of x_0 extends to $\pm\infty$, or there is a critical orbit of f, whose orbit is attracted to the orbit of x_0.

The proof of this theorem may be found in [24].

The Mandelbrot set is defined by the behavior of the critical point for the function f_c. Clearly, there is only one critical point $z=0$ for the polynomial f_c. The subset of the parameter plane (or c-plane) consists of all parameter value c for which, the orbit of $z=0$ under the map f_c, i.e.

$$f_c : 0 \to c \to c^2 + c \to \left(c^2 + c\right)^2 + c \to \cdots$$

is bounded is termed as the Mandelbrot set.

Note that there are infinitely many Julia sets (one for each c), whereas, there is only one Mandelbrot set.

Definition 2.14 [24]

The Mandelbrot set M is defined as:

$$M = \{c \in C : \textit{The Orbit of } 0 \textit{ is bounded under iteration by } f_c\}$$

$$= \left\{c \in C : \exists r > 0, \left| f_c^n(0)\right| \le r,\ \forall n \in N \right\}$$

3. GEOMETRICAL STRUCTURE OF THE MANDELBROT SET AND DIFFERENT TERMS RELATED TO THE STRUCTURE

One of the particular interests is to represent Mandelbrot set, graphically. The simplest algorithm for generating a representation of the Mandelbrot set is that of Julia set, where we color each point on the parameter space, depending on where its attractor lies i.e. whether it is attracted to infinity or bounded within the set. The Mandelbrot set's true visual beauty relies on the coloring near its boundaries. Developing a strong coloring algorithm helps display the beauty of the set, by providing the stunning visual aspect of the set which also gives the excitement of studying the set. One of the most popular ways of doing this is by assigning different colors to the points in the various regions such as inside the set boundary of the set. Also, for the points just outside the boundary, colors are determined by the number of iterations needed by the point to exceed a certain test value (usually 2). In the Figure 3.1, we use blue for the points inside the set, green for that in the boundary of the set and orange color for the points just outside the set.

Gradually, deeper color in orange indicates less number of iterations needed to exceed test value 2 in magnitude.

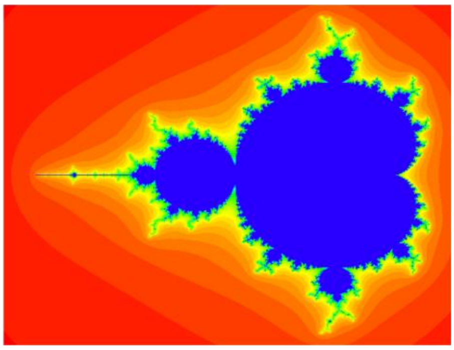


Figure 3.1: (The Mandelbrot Set)

Computer images of the Mandelbrot set, as in the figure 3.1, shows the estimated geometry of the Mandelbrot set. It contains a big cardioids shaped region, called the *body* of the Mandelbrot set. This region is indicated by B in *Figure 3.2* and it intersects the real axis at $c = \frac{1}{4}$ & $c = -\frac{3}{4}$. Towards its left, a circular area H with center at $c = -1$ and radius $\frac{1}{4}$ is attached, called the *head* of the set. The surface of these two parts is covered by some richly detailed structure of decoration, which makes the set a fractal one. Closer inspection of these decorations shows that all of them are different in shape. Any such decoration directly attached to the body is called a *primary bulb* or decoration. In turn, there are many smaller decorations attached to the boundary of each of these decorations as *antennas*. Again, antennas attached to each decoration seem to consist of several *spokes*. The number of such spokes varies from decoration to decoration as clearly visible in the *Figure 3.2*. Towards the left of head (just touching it), another circular region (smaller than the head) is attached, called the *secondary head*. It is important to mentioned here that though the Mandelbrot set is a fractal object, its boundary is so complex and intricate that it has an integer dimension two [30].

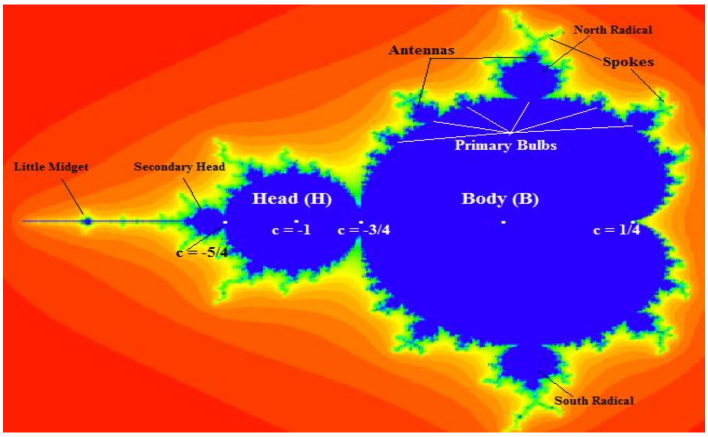


Figure 3.2: (Components of Mandelbrot)

4. SOME PROPERTIES OF MANDELBROT SET

In this section, we have studied some properties of the Mandelbrot sets for the complex polynomial of the form $z^2 + c$.

Proposition 4.1

All the points within the circle $S_{\frac{1}{4}}(0)$ of radius $\frac{1}{4}$ and centre at origin are contained in the Mandelbrot set.

Proof

Suppose, $c \in S_{\frac{1}{4}}(0) \Rightarrow |c| \leq \frac{1}{4}$.

Recall that the orbit of $'0'$ under f_c is:

$$0,\ c,\ c^2, (c^2 + c)^2 + c,\ \{(c^2 + c)^2 + c\}^2 + c,\ \cdots$$

We apply induction to show that the orbit of $'0'$ under f_c is bounded.

Clearly, $|f_c(0)| = |c| \leq \frac{1}{4}$

$$|f_c^2(0)| = |f_c(c)| = |c^2 + c| \leq |c^2| + |c| \leq \left(\frac{1}{4}\right)^2 + \frac{1}{4} < \frac{1}{2}$$

Suppose, $|f_c^r| < \frac{1}{2}$ where, r is any natural number.

Now, $|f_c^{r+1}(0)| = |f_c(f_c^r(0))| = |\{f_c^r(0)\}^2 + c| \leq |f_c^r|^2 + |c| \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

By law of induction, the orbit of $'0'$ under f_c is bounded and hence $c \in M$.

Proposition 4.2

The Mandelbrot set M is bounded.

Proof

We have $|f_c(0)| = |c|$

$$|f_c^2(0)| = |f_c(c)| = |c^2 + c| \geq |c|^2 - |c| = |c|(|c| - 1)$$

For $|c| > 2$, we have $\delta = |c| - 1 > 1$, and so $|f_c(c)| \geq |c|\,\delta$.

$$|f_c^3(0)| = |f_c(f_c(c))| \geq |f_c(c)|\,\delta \geq |c|\delta^2$$

Suppose, $|f_c^m(0)| \geq |c|\delta^{m-1}$ for some $m \in N$

Now, $|f_c^{m+1}(0)| = |f_c(f_c^m(0))| \geq |f_c^m(0)|\,\delta \geq |c|\,\delta^m$

By induction, $\left| f_c^n(0) \right| \geq |c| \delta^{n-1}$

Thus for $|c| > 2$ the iterates of '0' under f_c diverse to infinity, and hence $c \notin M$. This shows that entire Mandelbrot set is contained in the circle of radius 2 and centre at the origin i.e.

$|c| \leq 2 \quad \forall\, c \in M$. Thus, the Mandelbrot set is bounded.

Proposition 4.3

The Mandelbrot set M is symmetric about the real axis.

Proof

Recall that any subset S of the complex plane is symmetric about the real axis, if conjugate of each element of S belongs to S i.e. $z \in S \Rightarrow \bar{z} \in S$.

First we show that $\overline{f_c^n(z)} = f_{\bar{c}}^n(z)$.

$$\overline{f_c(z)} = \overline{z^2 + c} = (\bar{z})^2 + \bar{c} = z^2 + \bar{c} = f_{\bar{c}}(z)$$

In order to apply induction we assume that $\overline{f_c^m(z)} = f_{\bar{c}}^m(z)$ for some $m \in N$.

$$\overline{f_c^{m+1}(z)} = \overline{f_c\left(f_c^m(z)\right)} = \overline{\left\{f_c^m(z)\right\}^2 + c} = \left\{\overline{f_c^m(z)}\right\}^2 + \bar{c} = \left\{f_{\bar{c}}^m(z)\right\}^2 + \bar{c} = f_{\bar{c}}^{m+1}(z)$$

Thus by induction, $f_c^n(z) = f_{\bar{c}}^n(z)$ for all $n \in N$ & $z \in C$.

Now, let $c \in M \Rightarrow \left\{\left| f_c^n(0) \right|\right\}_{n=1}^{\infty}$ is bounded.

$\Rightarrow \left\{\left| \overline{f_c^n(0)} \right|\right\}_{n=1}^{\infty}$ is bounded as $|z| = |\bar{z}|$

$\Rightarrow \left\{\left| f_{\bar{c}}^n(0) \right|\right\}_{n=1}^{\infty}$ is bounded

$\Rightarrow \bar{c} \in M$

Hence the set M is symmetric about the real axis.

Proposition 4.4

The Mandelbrot set M intersect the real line on the closed interval $\left[-2, \frac{1}{4}\right]$ i.e. $R \cap M = \left[-2, \frac{1}{4}\right]$.

Proof

The orbit of 0 under f_{-2} is: $0, -2, 2, 2, 2, \cdots$ which is bounded so $-2 \in M$.

Also, by proposition 4.2, the Mandelbrot set is contained in the circle of radius 2 and center at the origin, therefore, no real number less than -2 belongs to M.

For $c > \frac{1}{4}$, we have for any $n \in N$

$$f_c^{n+1}(0) - f_c^n(0) = f_c\left(f_c^n(0)\right) - f_c^n(0)$$

$$= \left[f_c^n(0)\right]^2 - f_c^n(0) + c$$

$$= \left[f_c^n(0) - \frac{1}{2}\right]^2 + c - \frac{1}{4}$$

$$\geq c - \frac{1}{4} > 0$$

Therefore, $f_c^{n+1}(0) > f_c^n(0) \quad \forall\ n \in N \Rightarrow f_c^n(0) \to \infty \quad as \quad n \to \infty$.

Hence, $c \notin M$.

The fixed points of f_c are given by:

$$f_c(z) = z \quad i.e., \quad z^2 - z + c = 0$$

Solving $z_1 = \frac{1}{2}\left\{1 - \sqrt{1-4c}\right\}, \quad z_2 = \frac{1}{2}\left\{1 + \sqrt{1-4c}\right\}$.

For $-2 < c < 0$ we have $1 < \sqrt{1-4c} < 3$ and therefore $-1 < z_1 < 0$ and $1 < z_2 < 2$ which implies that $0 \in [z_1, z_2]$. In this case $f_c^2[z_1, z_2] = f_c[c, z_2] \subseteq [c, z_2]$. It follows that the orbit of ' 0 ' under f_c is bounded, so $c \in M$.

For $0 \leq c \leq \frac{1}{4}$, we have $0 \leq z_1 \leq \frac{1}{2}$ and $\frac{1}{2} \leq z_2 \leq 1$.

In this case, $f_c[0, z_2] = [c, z_2] \subseteq [0, z_2]$ so in this case also the orbit of ' 0 ' under f_c is bounded and hence $c \in M$.

Combining all these one can easily conclude that $M \cap R = \left[-2\,,\,\frac{1}{4}\right]$.

Proposition 4.5

For any $c \in C$, $c \in M$ if and only if the Julia set J_c of f_c is connected.

The main component of the Mandelbrot set are loops generated by the periodic fixed points of f_c.

A loop is a smooth (differentiable), closed and simple curve in the complex plane. Therefore, before proving the theorem, we first prove the following lemma about loops.

Lemma 4.5.1

Let L be a loop in the complex plane and $f_c(z) = z^2 + c$, then

a) If c is in the interior of L, then $f_c^{-1}(L)$ is a loop, and the inverse image of the interior of L is the interior of $f_c^{-1}(L)$.

b) If c lies on L, then $f_c^{-1}(L)$ is a figure eight (a curve in the shape 8) with self intersection at 0 such that the inverse image of the interior of L is the interior of the two loops.

Proof

First of all, we prove the lemma for a circle in the complex plane. The reason behind is that, by applying continuous transformation a circle can be transformed into a closed loop that we want. So, if we prove the lemma for a circle, we prove it for any closed loop in the complex plane.

(a)

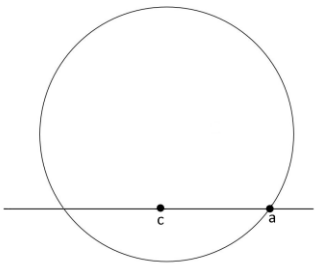


Figure 4.1

Suppose c lie inside the circle L. We can translate the circle such that c is the origin as shown in the *figure 4.1*.

Since now $c = 0$, $f_c(z) = z^2 \Rightarrow f_c^{-1}(z) = \pm\sqrt{z}$.

Let $w = re^{i\theta}$ be a point on L, then $f_c^{-1}(z) = \pm\sqrt{r}\, e^{i\frac{\theta}{2}}$. Start with w as the point a on the x-axis and consider only the positive square root of the inverse function, $f_c^{-1}(w)$ traces a loop in the complex plane as w travels around the circle. Likewise, the point closest to c is when w is at the angle $\frac{3\pi}{2}$ so, the inverse function is closest to the origin at an angle of $\frac{3\pi}{4}$. Therefore, doing a full 2π rotation around the circle, the inverse function maps out a rotation of π around the new loop and after a full 2π rotation we are the same distance away from c as when we started. So the inverse function at an angle of π will be at $-\sqrt{w}$. Thus tracing out the first 2π radians will give us a curve as shown in figure 4.2.

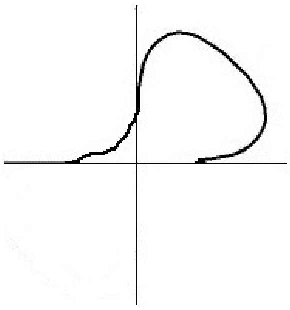


Figure 4.2

However, as we go around the loops, a second loop is drawn at the same time on taking the negative square root. The negative inverse function will map out a loop, exactly as the positive inverse function, only reflected about the origin. Putting all these pieces together, we should get a loop that looks like *figure 4.3*.

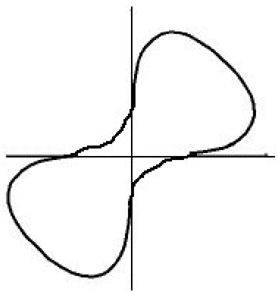


Figure 4.3

(b) Suppose c is the Origin Lie in the Circumference of the Circle L as given in Figure 4.4

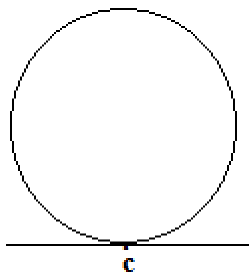


Figure 4.4

Consider first the positive square roots for the inverse function and suppose the point w will move around the circle from its initial position c. The furthest point from c for the inverse function will be at an angle $\theta = \frac{\pi}{4}$, since the farthest point from c on the circle is at an angle $\frac{\pi}{2}$. Again the point w will make a full rotation around the circle and returns to its initial position c when it is traced out an angle π , therefore, for the inverse function, the loop returns to the origin when it traced out an angle $\frac{\pi}{2}$. Also, by taking the negative square root function, another loop is formed as the mirror image of the first loop. These two loops are connected at the origin and hence they formed a figure eight as shown in the *figure 4.5*.

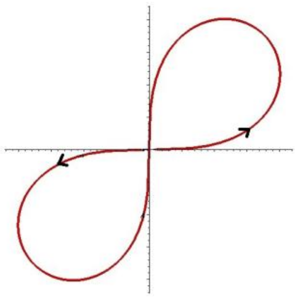


Figure 4.5

Proof of the Proposition 4.5

First suppose that $c \in M \Rightarrow \left\{ f_c^{\,k}(0) \right\}$ is bounded. Let L be a large circle in C which contains all the points of $\left\{ f_c^{\,k}(0)\right\}$. As $f_c(0) = c$, so c is in the interior of L. By lemma (a), $f_c^{-1}(L)$ is a loop where the interior of L is mapped to the interior of $f_c^{-1}(L)$. Again $f_c^{\,2}(0) = c^2 + c$ is in the interior of L, so by applying the same lemma again, $f_c^{-2}(L)$ is in the interior of $f_c^{-1}(L)$. Continuing in this way, one may have the loop $\left\{ f_c^{-k}(0)\right\}$, inside a loop $\left\{f_c^{-k+1}(0)\right\}$ which is again inside the another loop $\left\{ f_c^{-k+2}(0)\right\}$ and so on. Suppose, K be the intersection of all such loops.

Now, $f_c^{\,k}(z) \to \infty, \quad \forall z \in \hat{C} - K$. Thus the basin of attraction of the fixed point at ∞ is the set $\hat{C} - K$, thus K is the filled in Julia set of f_c, i.e., $\partial A(\infty) = \partial(K) = J_c$ [here $\partial(K)$ represent the boundary of the set K]. Note that, K is the intersection of infinitely many closed and simply connected set. So, K itself is closed and simply connected by a theorem of topology. Thus, the boundary $\partial(K)$ of K is simply connected and hence the Julia set J_c of f_c is connected.

On the contrary, suppose that the Julia set J_c of f_c is connected. If possible let, $\left\{ f_c^{\,k}(0)\right\}$ is not bounded. As above, consider a large circle L in $\hat{C}$ such that all points out side L go to ∞ . Note that, by our hypothesis $0 \to \infty$ under iteration of f_c even though $0 \in L$. Suppose, there exists $n \in N$ such that for $\left\{f_c^{\,k}(0) : k \le n\right\}$ lie inside L and $\left\{f_c^{\,k} : k > n\right\}$ is in the outside of L. As before, create a sequence of loops, where, each new loop is inside the previous loop. Now,

$$f_c^{\,n}(0) \in L \Rightarrow f_c^{\,n-1}(f_c(0)) = f_c^{\,n-1}(c) \in L \Rightarrow c \in f_c^{\,1-n}(L)$$

So by applying lemma *4.5.1 (b)* we will create the loops in a unique way. By this lemma, $f_c^{-1}(L)$ is a figure eight, i.e., an 8 shaped double loop intersecting at the origin. Applying the same lemma again, $f_c^{-2}(L)$ is a figure eight, inside the figure eight created by $f_c^{-1}(L)$, and intersects itself at origin. Keep doing this n times, one can get n numbers of figure eight each of which intersect at the origin.

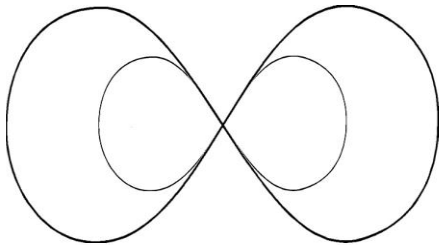


Figure 4.6

Suppose, N be the smallest of all such figure eight. Since the Julia set J_c of f_c is invariant under iteration of f_c i.e., $f_c(J_c) = J_c$, it must contain the smallest of all the figure eight, which is N. Since the Julia set is symmetric about the origin, J_c must lie on both sides of the figure eight. Now, $0 \notin J_c$ as $0 \to \infty$. Therefore, 0 disconnect the J_c inside N which is absurd as by our assumption J_c is connected. Hence, $\left\{ f_c^{\,k}(0) \right\}$ is bounded.

Note, this proposition clearly suggests that the Julia sets corresponding to the points outside the Mandelbrot set will be disconnected one or Cantor dust. The link goes much deeper than that, the Julia sets corresponding to different areas of the Mandelbrot set have very different structures. Figure 4.7 shows such Julia sets for some values of $'c'$ belonging to the different areas of the Mandelbrot set.

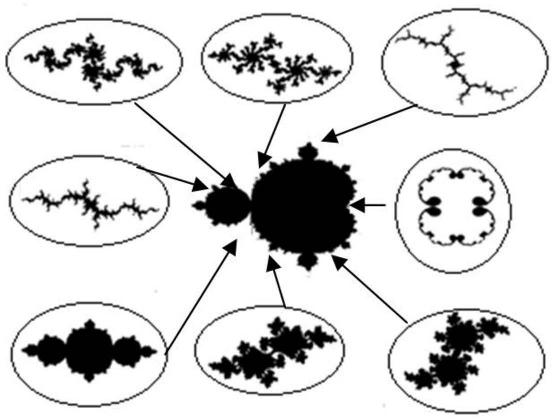


Figure 4.7: (Julia Sets Corresponding to 'c' Located at Different Areas of Mandelbrot Set)

Proposition 4.6

If $f_c(z) = z^2 + c$ has an attracting periodic point, then c is on the Mandelbrot set M.

Proof

Suppose that f_c has an attracting periodic point, say z_0,

The Schwarzian derivative of f_c:

$$Sf_c(z) = \frac{D^3 f_c(z)}{Df_c(z)} - \frac{3}{2}\left[\frac{D^2 f_c(z)}{Df_c(z)}\right]^2 < 0 \text{ i.e., } f_c \text{ is of negative Schwarzian derivative.}$$

Clearly, $Df_c(z) = 0 \Rightarrow z = 0$ .i.e., $z = 0$ is the only critical point of f_c. Therefore, by the Theorem 2.2, the iterates of the critical point of f_c, i.e., $z = 0$ converges to z_0 and its iterates. Hence, the orbit of $z = 0$ under f_c is bounded, which implies that c is on the Mandelbrot set M .

Proposition 4.7

The body B of the Mandelbrot set M is the set of those complex numbers c for which f_c has an attracting or a neutral fixed point.

Proof

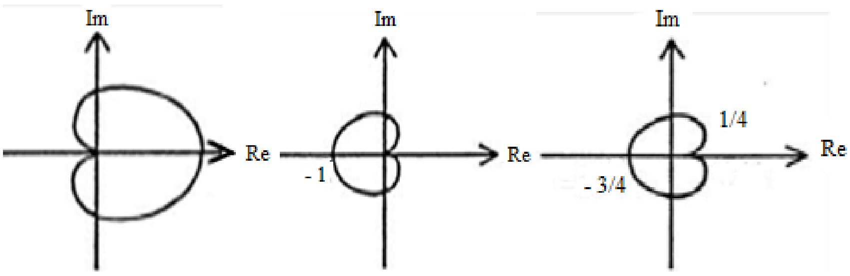
By Proposition 4.6, if f_c has an attracting fixed point then $c \in M$. The fixed points of f_c are:

$$z_1 = \frac{1}{2}\left\{1 - \sqrt{1-4c}\right\} \text{ and } z_2 = \frac{1}{2}\left\{1 + \sqrt{1-4c}\right\}.$$

Since, $Df_c(z) = 2z$ so f_c has either an attracting or a neutral fixed point if

$$|Df_c(z_1)| \le 1 \text{ or } |Df_c(z_2)| \le 1 \text{ i.e., } \left|1 - \sqrt{1-4c}\right| \le 1 \text{ or } \left|1 + \sqrt{1-4c}\right| \le 1.$$

Consider the complex number $w = \sqrt{1 - 4c}$



(a) **(b)** **(c)**

Figure 4.8

Suppose that, $w = x + i\,y$ and its polar coordinates are (r, θ) i.e., $r^2 = x^2 + y^2$ and $\cos\theta = \frac{x}{r}$.

Now, $$|1 - w| = 1 \Rightarrow x^2 + y^2 = 2x \Rightarrow r = 2\frac{x}{r} \Rightarrow r = 2\cos\theta \Rightarrow r^2 = 4\cos^2\theta = 2 + 2\cos(2\theta)$$

Therefore, polar coordinates of w^2 are $(r^2, 2\theta)$ i.e. $(2 + 2\cos(2\theta), 2\theta)$, which shows that w^2 has polar coordinates (r, θ) such that $r = 2 + 2\cos(2\theta)$, $\theta \in [0, 2\pi]$.

Clearly, this is an equation of cardioid which meets the real axis when $\theta = 0$ and $\theta = \pi$ i.e. at $(4, 0)$ and $(0, 0)$. This cardioid is represented in *Figure 4.8 (a).*

Again, $|1+w|=1 \Rightarrow x^2+y^2=-2x \Rightarrow r=-2\cos\theta \Rightarrow r^2=2+2\cos(2\theta)$. Thus a similar conclusion can be drawn for w satisfying $|1+w|=1$.

From the above discussion it is clear that the locus of the complex number w^2 satisfying $|1-w|=1$ or $|1+w|=1$ is the cardioid $r=2(1+\cos\theta)$, $\theta\in[0,2\pi]$. Therefore, the locus of the complex numbers $-\frac{w^2}{4}$ satisfying the same conditions as above is the cardioids formed by reflecting the cardioids in *Figure 4.8 (a)* across the origin and contracting by a factor $\frac{1}{4}$. This cardioid is represented in the *Figure 4.8 (b*) which meets real axis at $(0,0)$ and $(-1,0)$. Finally, the locus of $\frac{1}{4}(1-w^2)$ is the cardioid obtained by shifting the origin to $\left(\frac{1}{4},0\right)$. Applying that shift to the cardioid in *Figure 4.8 (b*), we get the cardioid in the *Figure 4.8 (c),* which represent the body B of the Mandelbrot set. As $w=\sqrt{1-4c} \Rightarrow c=\frac{1}{4}(1-w^2)$, we conclude that the complex numbers c for which f_c has an attracting fixed point comprises the body B of the Mandelbrot set.

Note that the neutral fixed points of f_c occupies the boundary of the cardioid represented in *Figure 4.8 (c)* i.e., the boundary of the body B of the Mandelbrot set as for these c, $Df_c(z_1)=1$ or $Df_c(z_2)=1$.

Proposition 4.8

The head H of the Mandelbrot set M is the set of those complex numbers $'c'$ for which f_c has an attracting 2-cycle.

Proof

The 2-cyles of f_c are given by -

$$
\begin{aligned}
f_c^2(z)=z \quad &\Rightarrow f_c(z^2+c)=z \\
&\Rightarrow (z^2+c)^2+c=z \\
&\Rightarrow z^4+2cz^2-z+c^2+c=0 \\
&\Rightarrow (z^2-z+c)(z^2+z+c+1)=0
\end{aligned}
$$

Since, $z^2-z+c=0$ will give the fixed points of f_c, so in order to disregard the fixed points we omit the factor (z^2-z+c) and get

$$z^2+z+c+1=0$$

Solving, we get the roots as $z_1=\frac{1}{2}\left(-1+\sqrt{-3-4c}\right)$ and $z_2=\frac{1}{2}\left(-1-\sqrt{-3-4c}\right)$.

Clearly $f_c(z_1) = z_2$ & $f_c(z_2) = z_1$.

Now $\{ z_1, z_2 \}$ will be an attracting 2-cycle of f_c if

$$\left| D f_c^2(z_1) \right| < 1 \Rightarrow \left| Df_c(f_c(z_1))\, Df_c(z_1) \right| < 1$$

$$\Rightarrow \left| Df_c(z_2) Df_c(z_1) \right| < 1$$

$$\Rightarrow 4 \left| z_1 . z_2 \right| < 1$$

$$\Rightarrow \left| 1 + c \right| < \frac{1}{4}$$

Thus, it is clear that all the complex numbers bounded by the circle $\left| c + 1 \right| = \frac{1}{4}$ of radius $\frac{1}{4}$ and center at $(0, -1)$ represent the attracting 2-cycle of f_c. This circle is the head H of the Mandelbrot set.

In addition to Proposition 4.7 and 4.8, all the primary bulbs in the Mandelbrot set can be put in a correspondence with the existence of a periodic orbit for a given period.

Substituting $Df_c(z) = 2z$ by $r e^{i\theta}$ where $r \geq 0$, $0 \leq \theta < 2\pi$ for the fixed points of f_c we get $c = \frac{1}{2} r e^{i\theta} - \frac{1}{4} r^2 e^{i2\theta}$.

As $r = 1$ for neutral fixed points of f_c for each value of θ $(0 \leq \theta < 2\pi)$, we get a complex number $c = \frac{1}{2} e^{i\theta} - \frac{1}{4} e^{i2\theta}$ which lies on the surface of the body 'B'. It turns out at that, the parameter values of $\theta = \frac{2\pi}{n}$, $n = 2, 3, 4, 5, \ldots$ one of the primary bulb is attached to the body 'B'.

Moreover, the bulb at $\theta = \frac{2\pi}{n}$ (for a particular $n = 2, 3, 4, \cdots$) is the set of those complex numbers c for which, f_c has an n-cycle. For example, when $n = 2$ i.e., $\theta = \pi$ we have the head H of the Mandelbrot set, which is the collection of all 2-cycles of f_c as shown in proposition 4.8.

Also, there is a surprising relationship between the cycle number 'n' and the number of spokes on the antennas of these primary bulb. It is observed that these two numbers are same for any 'c' inside a primary bulb. In figure 4.9 we have summarized some of these results graphically.

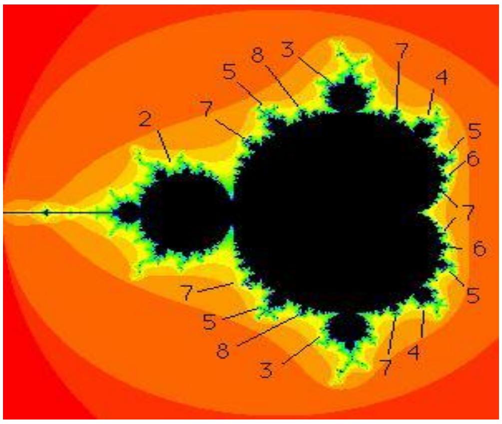


Figure 4.9: (Some Primary Bulbs, Labeled According to Their Period)

There are sevaral amazing features of the Mandelbrot set in relation to the periodic numbers of bulbs which are scattered in the surface of the Mandelbrot set. First, it is observed that, if we travel clockwise from the head H (whose periodic number is 2), the next big bud (smaller than H) attached to the body B corresponds to period-3 behavior, after that we get next big bud with house parameters belonging to attractive cycle of period 4, and then the next big bud is of period-5 and so on. *Figure 4.10* illustrat this fact graphically.

Another amazing fact is the presence of Fibonacci sequence in the Mandelbrot set. It is observed that, if two neighbouring buds have periodicity n_1 and n_2 then the periodicity of the largest bud that is smaller than these two and lies in between is $n_1 + n_2$. The famous Fibonacci sequence $\langle F(n) \rangle$ is defined as $F(0) = 1 = F(1),\ F(n) = F(n-1) + F(n-2)\,;\ n = 2, 3, 4, \cdots.$

We start with $F(2)$ to be the periodicity of the period two bulb (i.e., the head H), and then taking $F(3)$ to be the period of the next largest bulb by moving clockwise around the boundary, $F(4)$ the period of the next largest bulb moving counter clockwise, and so on. Note that, here we always consider the largest bulb between the previous two, and if at any instance we move clockwise, then the next motion will be counter clockwise.

This observation is exibited graphically in *figure 4.11*.

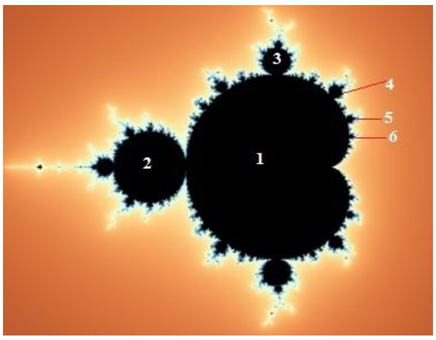


Figure 4.10: (Ordering of Periodicity of the Bulbs in Natural Numbers)

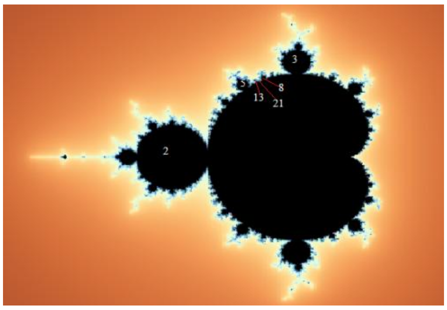


Figure 4.11: (Fibinacci Sequence in the Mandelbrot Set)

Further, to explore the geometry of the periodic orbits in the primary bulbs of the Mandelbrot set, we assign each bulb a real number of the form $\frac{m}{n}$, called the rotation number of the bulb.

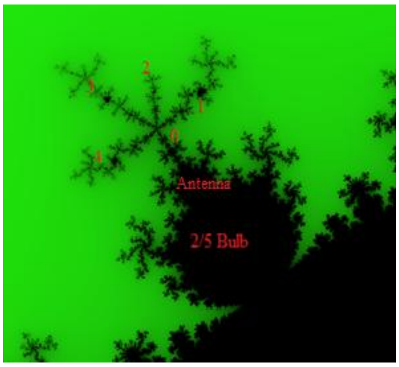
Definition 4.1 [1]

The *rotation number* of a bulb in the Mandelbrot set is the real number $\frac{m}{n}$ where, the denominator $'n'$ is the periodicity of the bulb and if $\{z_1, z_2, \cdots, z_n\}$ is a periodic orbit on that bulb, then the numerator $'m'$ is the number such that

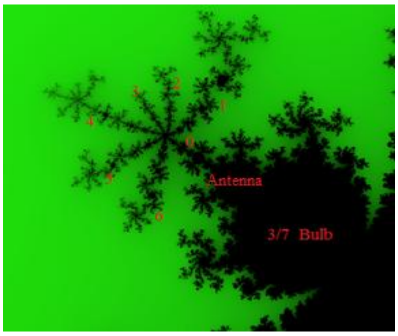
$$f_c(z_k) = z_{k+m} \quad , \quad k = 1, 2, \cdots, n$$

Note that the bulb with rotation number $\frac{m}{n}$ is termed as $\frac{m}{n}$ bulb.

One can easily determine the rotation number of a bulb by closer inspection at the bulb in the Mandelbrot set [10]. The total number of spokes in each of the antenna that are attached to the bulb is the same and it determines the periodic number of the bulb i.e. the denominator *n* of the rotation number. These spokes are different in length. The shortest one counterclockwise from the main one is the *m* th one. These facts are illustrated in *figure 4.12* by considering two bulbs with rotation numbers $\frac{2}{5}$ and $\frac{3}{7}$.



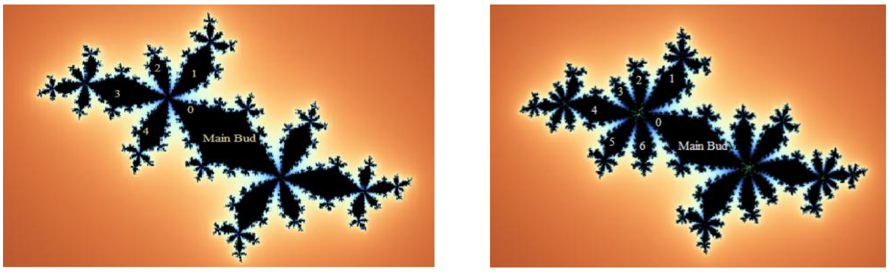
(a)



(b)

Figure 4.12: [Bulbs with Rotation Number (a) 2/5 and (b) 3/7]

Julia sets, corresponding to the parameter value belongs to such bulb consists of several buds meeting at a point. The number of such buds for each parameter value within a bulb is the same, and this number of such buds gives the periodic number *n* of the bulb. The smallest of these *n* buds counterclockwise from the main bud (excluding the main bud) is the *m* th one. In this way, the rotation number can be determined from the Julia set of the appropriate value of *c*. This fact is demonstrated graphically in figure 4.*13*.



(a) **(b)**

Figure 4.13: [Julia Set with Parameter Value (a) in the 2/5 Bulb (b) in the 3/7 Bulb]

The rotation numbers of the primary bulb has an interesting relation with the Farey sequence, which is relevant in number theory. In order to understand the relation, we first explore the concept of Farey addition. The Farey addition of two rational numbers $\frac{m_1}{n_1}$ and $\frac{m_2}{n_2}$ is defined as

$$\frac{m_1}{n_1} \oplus \frac{m_2}{n_2} = \frac{m_1+m_2}{n_1+n_2}$$

It is an established fact that, if the two fractions have no number with a smaller denominator than either of their denominators between them, then the resulting fraction is the fraction with the smallest denominator between them [15]. For better understanding of Farey addition process, we construct the diagram, given in figure 4.*14*, as follows:

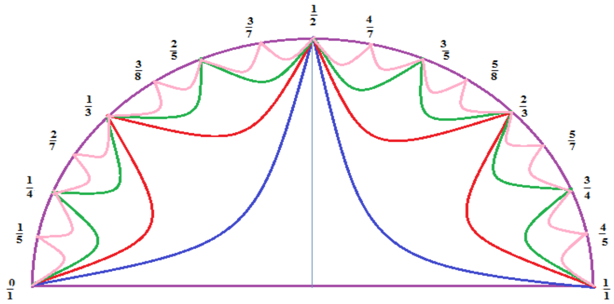


Figure 4.1: [The Farey Diagram]

First, we draw a semi-circle and considering the centre as origin, we denote the point in the semi-circle at π radians by $\frac{0}{1}$, at 0 radians by $\frac{1}{1}$, $\frac{\pi}{2}$ radians by $\frac{1}{2}$.

Now draw curved arcs joining $\frac{0}{1}$ to $\frac{1}{2}$ and $\frac{1}{1}$ to $\frac{1}{2}$. We say that $\frac{1}{2}$ is the *Farey child* of the *Farey parents* $\frac{0}{1}$ and $\frac{1}{1}$. Similarly, $\frac{2}{3}$ is the Farey child of the Farey parents $\frac{1}{2}$ and $\frac{1}{1}$; $\frac{1}{3}$ is the Farey child of $\frac{0}{1}$ and $\frac{1}{2}$ and so on. Note that the Farey child is nothing but the Farey addition of their respective Farey parents.

The terminology Farey child and the Farey parents are first introduced by Devaney [11].

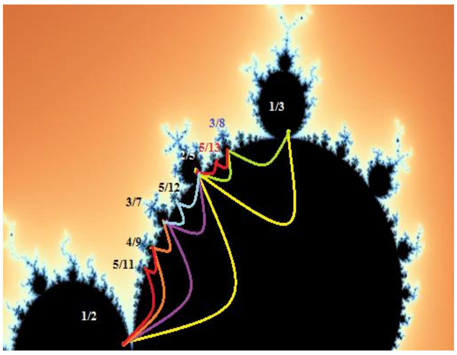


Figure 4.15: [Rotation Number of Primary Bulb and Farey Addition]

It is observed that the primary bulbs in the Mandelbrot set are arranged in a similar fashion of Farey addition method, in terms of their rotation numbers. The rotation number of the biggest primary bulb in between two bulbs is the Farey child of their rotation numbers. For example, the rotation number of the biggest bulb in between the bulbs of rotation number $\frac{1}{2}$ and $\frac{1}{3}$ is their Farey child $\frac{2}{5}$.

Similarly, the rotation number of the biggest bulb in between $\frac{2}{5}$-bulb and $\frac{1}{3}$-bulb is their Farey child $\frac{3}{8}$ and so on. We have explicitly exhibited this fact in figure 4.15.

5. CONCLUSION

The dynamic behavior and graphical complexity of Mandelbrot set provides a beautiful example of the fascinating world of fractals. Every little piece of it is loaded with some mathematical meaning. Our study helped us to understand some of such mathematical properties. We observed some amazing features shown by the periodic numbers and rotation numbers related to the primary bulbs of the Mandelbrot set, which are guided by sequences like Fibonacci sequence, Farey sequence etc. We think, there is enough scope of further investigation to find out many such amazing features in the Mandelbrot set.

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